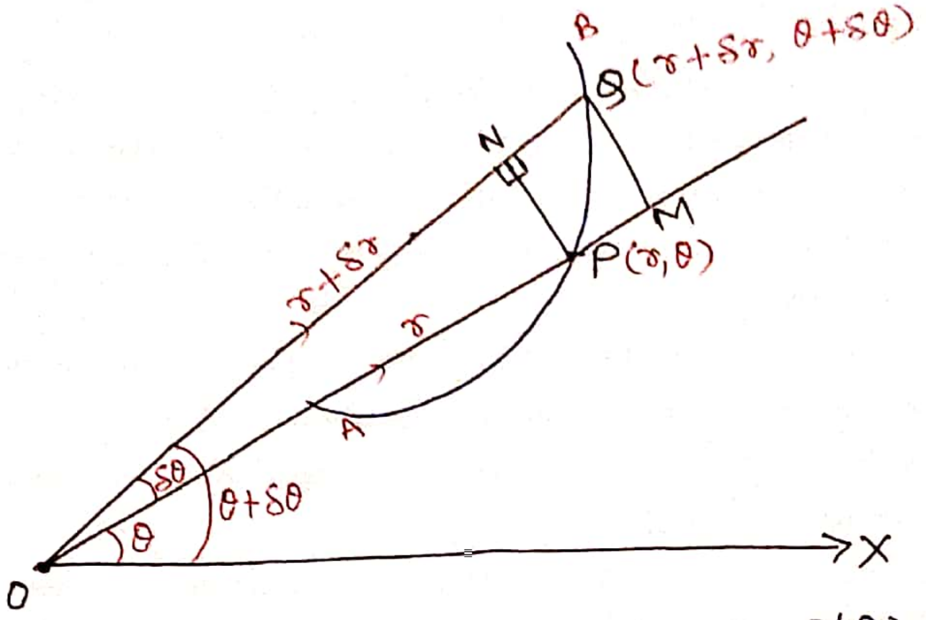
VELOCITY AND ACCELERATION IN POLAR FORM

Expression for velocity along and perpendicular to the Radius vector

OR

Radial and Transverse velocity.



Let AB be the polar curve $r = f(\theta)$.
OX be the initial line.
Let P and Q be any two positions of a moving particle at time 't' and 't+δt' resp.

Join O-P and O-Q.
and draw QM ⊥r on OP produced
and PN ⊥r on OQ.

Clearly, $OP = r$, $OQ = r + \delta r$
$\angle POX = \theta$, $\angle QOX = \theta + \delta\theta$.

So, $\angle POQ = \theta + \delta\theta - \theta$
$= \delta\theta$.

Now,

In ΔOMQ

$$\sin \delta\theta = \frac{QM}{OQ}$$

$$\Rightarrow \quad QM = OQ \sin \delta\theta$$
$$= (r+\delta r) \sin \delta\theta$$
$$= (r+\delta r) \cdot \delta\theta \quad \Big| \because \delta\theta \text{ is very small } \sin\delta\theta = \delta\theta.$$

$$\cos \delta\theta = \frac{OM}{OQ}$$

$$\Rightarrow \quad OM = OQ \cos\delta\theta$$
$$= (r+\delta r)\cos\delta\theta$$
$$= (r+\delta r) \cdot 1 \quad \Big| \because \delta\theta \text{ is very small } \cos\delta\theta = 1.$$

Radial velocity along the radius vector OP.

= Rate of change of displacement along OP.

$$= \lim_{\delta t \to 0} \left(\frac{OM - OP}{\delta t} \right)$$

$$= \lim_{\delta t \to 0} \frac{(r+\delta r) - r}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{\delta r}{\delta t}$$

$$= \frac{dr}{dt}$$

$$= \dot{r}$$

Transverse velocity

= Rate of change of displacement ⊥r to the radius vector.

$$= \lim_{\delta t \to 0} \frac{QM}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{(r+\delta r)\delta\theta}{\delta t}$$

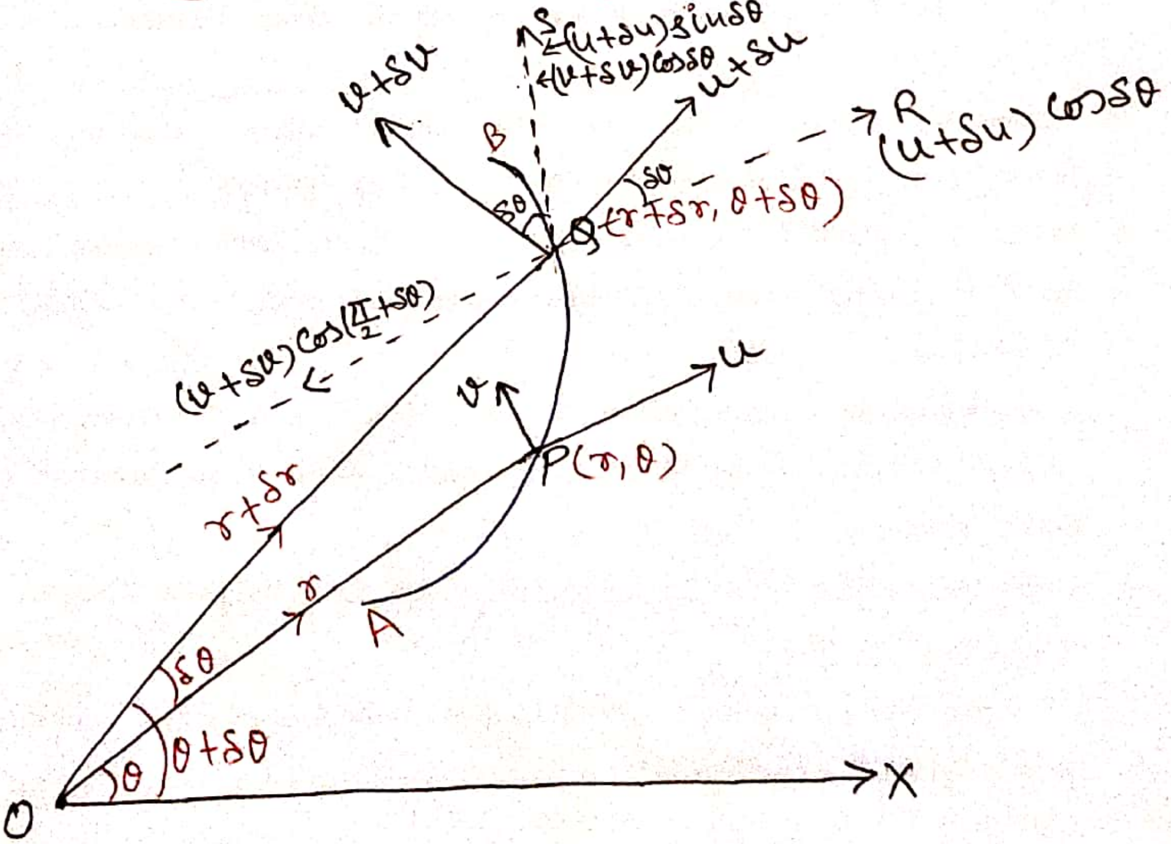
$$= \lim_{\delta t \to 0} \left[r \cdot \frac{\delta\theta}{\delta t} + \frac{\delta r}{\delta t} \cdot \delta\theta \right]$$

$$= \lim_{\delta t \to 0} \left[r \frac{\delta\theta}{\delta t} + \frac{\delta r}{\delta t} \cdot \frac{\delta\theta}{\delta t} \cdot \delta\theta \right]$$

$$= r \frac{d\theta}{dt} + 0$$

$$= r\dot{\theta}$$

RADIAL AND TRANSVERSE ACCELERATION



Let AB be the polar curve $r = f(\theta)$

OX be the initial line.

Let P and Q be the positions of moving particle at time t and $t+\delta t$ resp.

Let radial and transverse velocities at P and Q be (u, v) and $(u+\delta u, v+\delta v)$ resp.

Join O-P and O-Q.

Then the components of velocity along QR are

$(u+\delta u)\cos\delta\theta$, $(v+\delta v)\cos\left(\frac{\pi}{2}+\delta\theta\right)$

i.e. $(u+\delta u)\cos\delta\theta$, $(v+\delta v)(-\sin\delta\theta)$

and components of velocity along QS are

$(u+\delta u)\cos\left(\frac{\pi}{2}-\delta\theta\right)$, $(v+\delta v)\cos\delta\theta$

i.e. $(u+\delta u)\sin\delta\theta$, $(v+\delta v)\cos\delta\theta$.

Therefore, radial accln is equal to the rate of change of velocity along the radius vector OP

$$= \lim_{\delta t \to 0} \frac{(\text{Vel. along OP at Q} - \text{Vel. along OP at P})}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{(u+\delta u)\cos\delta\theta + (v+\delta v)(-\sin\delta\theta) - u}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{(u+\delta u) + (v+\delta v)(-\delta\theta) - u}{\delta t}$$

$\left[\because \delta\theta\right.$ is very small
$\cos\delta\theta = 1$ and $\sin\delta\theta = \delta\theta$.

$$= \lim_{\delta t \to 0} \frac{\cancel{u} + \delta u - v\,\delta\theta - \delta v\,\delta\theta - \cancel{u}}{\delta t}$$

$$= \frac{du}{dt} - v\frac{d\theta}{dt} - \lim_{\delta t \to 0} \frac{\delta v}{\delta t} \cdot \frac{\delta\theta}{\delta t} \cdot \delta t$$

$$= \frac{du}{dt} - v\frac{d\theta}{dt} - 0$$

$$= \frac{d}{dt}\left(\frac{dr}{dt}\right) - r\frac{d\theta}{dt}\left(\frac{d\theta}{dt}\right)$$

$$= \frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$$

$$= \ddot{r} - r(\dot{\theta})^2$$

Transverse Accl^n.

= Rate of change of velocity $\perp$r to the radius vector OP.

$$= \lim_{\delta t \to 0} \frac{(\text{Vel. } \perp r \text{ OP at } Q - \text{Vel. } \perp r \text{ OP at } P)}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{(u+\delta u)\sin\delta\theta + (v+\delta v)\cos\delta\theta - v}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{(u+\delta u)\delta\theta + (v+\delta v)\cdot 1 - v}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{u\,\delta\theta + \delta u\,\delta\theta + \cancel{v} + \delta v - \cancel{v}}{\delta t}$$

$$= u\frac{d\theta}{dt} + \lim_{\delta t \to 0} \frac{\delta u}{\delta t}\frac{\delta\theta}{\delta t}\cdot\delta t + \frac{d\theta}{dt}\frac{dv}{dt}$$

$$= u\frac{d\theta}{dt} + 0 + \frac{dv}{dt}$$

$$= \frac{dr}{dt}\frac{d\theta}{dt} + \frac{dv}{dt}$$

$$= \frac{dr}{dt}\cdot\frac{d\theta}{dt} + \frac{d}{dt}\left(r\frac{d\theta}{dt}\right)$$

$$= \frac{dr}{dt}\cdot\frac{d\theta}{dt} + \frac{dr}{dt}\cdot\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}$$

$$= 2\cdot\frac{dr}{dt}\cdot\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}$$

$$= 2\dot{r}\dot{\theta} + r\ddot{\theta}$$

$$= \frac{1}{r}\left(2r\cdot\frac{dr}{dt}\cdot\frac{d\theta}{dt} + r^2\frac{d^2\theta}{dt^2}\right)$$

$$= \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right).$$

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